

Chapter One

Complex Numbers

The complex numbers were invented for purely mathematical reasons, just like the real numbers and were intended to make things neat and tidy in solving equations. They were regarded with deep suspicion by the more conservative folk for a century. Complex numbers are points in the plane, together with a rule telling you how to multiply them. They are two-dimensional, whereas the real numbers are one dimensional.

Equations of the form $3x^2 + 5$ has no solution on the set of real numbers. Therefore, the set of complex numbers permits us to solve such like equations.

Definition 1: The set of complex numbers is denoted by $\mathbb{C}$ and is described by $\mathbb{C} = \{z / z = x + iy, x, y \in \mathbb{R} \text{ and } i^2 = -1\}$.

From the expression $z = x + iy$, x is called the real part and is denoted by $\text{Re}(Z)$

y is called the imaginary part and is denoted by $\text{Im}(Z)$.

Note: If $x = 0$, the number is called purely imaginary and if $y = 0$, the number is called purely real.

Complex numbers can be defined as an order pair (x, y) of real numbers that can be interpreted as points in the complex plane (z - plane) with coordinates x and y .

Example 1: Find the real & imaginary part of the following complex numbers :

a) $z = 3 + 7i$

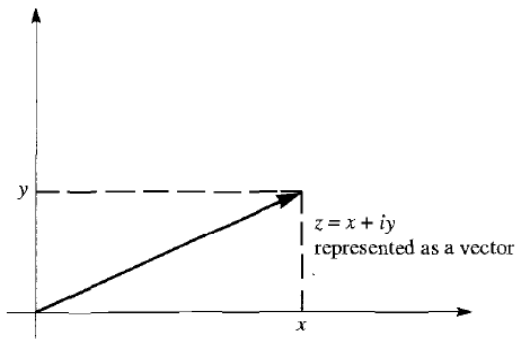
Solution : real part = 3 & imaginary part = 7

b) $z = 1 - i$

Solution : real part = 1 & imaginary part = -1

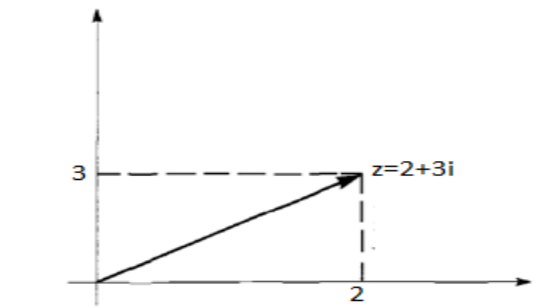
1.1. Plotting complex numbers

Any complex number $z = x + iy$ can be drawn in the complex plane as below :



Example 2: Draw the complex number $z = 2+3i$

Solution:



Equality of Complex numbers

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal iff $a = c$ & $b = d$.

Example 3 If $z_1 = 2 + ix$ and $z_2 = y + 6i$ are equal, then find the value of x & y .

Solution :

$$x = 6, y = 2.$$

1.2. Operations on Complex numbers

Let $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then

$$i) \quad z_1 + z_2 = (a + c) + i(b + d)$$

$$ii) \quad z_1 - z_2 = (a - c) + i(b - d)$$

$$iii) \quad z_1 \cdot z_2 = (a + ib) \cdot (c + id) = a(c + id) + ib(c + id) = ac + iad + ibc - bd = (ac - bd) + i(ad + bc)$$

$$iv) \quad \frac{z_1}{z_2} = \frac{(a + ib)}{(c + id)}, \quad z_2 \neq 0.$$

Example 4

If $z_1 = 2 + 3i$ and $z_2 = 4 + i$, then find a) $z_1 + z_2$ b) $z_1 - z_2$ c) $z_1 \cdot z_2$ d) $\frac{z_1}{z_2}$

$$\text{Sol : a) } z_1 + z_2 = 6 + 4i$$

$$b) \quad z_1 - z_2 = -2 + 2i$$

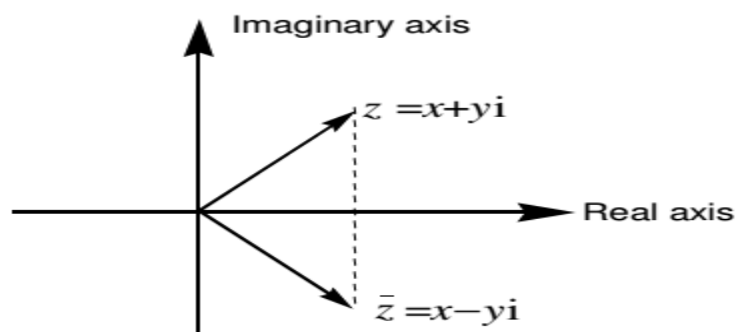
$$c) \quad z_1 \cdot z_2 = (2 + 3i) \cdot (4 + i) = 8 + 2i + 12i - 3 = 5 + 14i$$

$$d) \quad \frac{z_1}{z_2} = \frac{2 + 3i}{4 + i}$$

1.3. Conjugate of a complex number

Definition 2:

The conjugate of a complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined as $\bar{z} = x - iy$. It can be represented by the point $(x, -y)$ which is the reflection of the point (x, y) about the x-axis.



Example 5: Find the conjugate of the complex number $z = 2 + 9i$.

Solution :

$$z = 2 + 9i$$

$$\Rightarrow \bar{z} = 2 - 9i$$

Properties of Conjugate

$$a. \bar{\bar{z}} = z$$

$$b. z + \bar{z} = 2x = 2 \operatorname{Re}(z) = 2 \left(\frac{z + \bar{z}}{2} \right)$$

$$c. z - \bar{z} = 2iy = 2i \operatorname{Im}(z)$$

$$d. \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$e. \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$f. \overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \quad g. \overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

proof : a) let $z = x + iy$

$$\Rightarrow \bar{z} = x - iy$$

$$\Rightarrow \bar{\bar{z}} = x + iy$$

$$\therefore \bar{\bar{z}} = z.$$

$$d) \text{ Let } z_1 = x_1 + iy_1 \text{ \& } z_2 = x_2 + iy_2$$

$$\Rightarrow \bar{z}_1 = x_1 - iy_1 \text{ \& } \bar{z}_2 = x_2 - iy_2$$

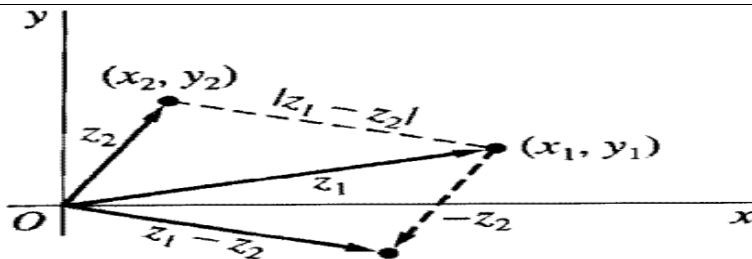
$$\begin{aligned} \text{Now } \overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)} = (x_1 + x_2) - i(y_1 + y_2) \\ &= x_1 - iy_1 + x_2 - iy_2 \\ &= \bar{z}_1 + \bar{z}_2 \end{aligned}$$

The others are left for the reader.

1.4. Modulus (Norm) of a complex number

Definition 2

The modulus of a complex number $z = x + iy$ is a non-negative real number denoted by $|z|$ and is defined as $|z| = \sqrt{x^2 + y^2}$. Geometrically, the number $|z|$ represents the distance between the point (x, y) and the origin.



Example 5: Find the modulus of the complex number $z = 3 - 4i$.

Solution : $z = 3 - 4i$

$$|z| = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Properties of modulus

a. $|z| = |\bar{z}|$

b. $|z|^2 = z \cdot \bar{z}$

c. $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

d. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$

e. $|z_1 + z_2| \leq |z_1| + |z_2|$ triangle inequality

f. $|z_1 - z_2| \geq |z_1| - |z_2|$

proof (a) let $z = x + iy$ from which $\bar{z} = x - iy$

$$\Rightarrow |z| = \sqrt{x^2 + y^2} \quad \Rightarrow |\bar{z}| = \sqrt{x^2 + y^2}$$

$$\therefore |z| = |\bar{z}|.$$

proof (b) let $z = x + iy$ from which $\bar{z} = x - iy$

$$\Rightarrow |z|^2 = x^2 + y^2$$

$$\text{Now } z \cdot \bar{z} = (x + iy) \cdot (x - iy) = x^2 + y^2$$

$$\therefore |z|^2 = z \cdot \bar{z}$$

$$\text{proof (c) } |z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 \cdot z_2}) = (z_1 \cdot z_2)(\bar{z}_1 \cdot \bar{z}_2) = z_1 \cdot \bar{z}_1 \cdot z_2 \cdot \bar{z}_2$$

$$= |z_1|^2 \cdot |z_2|^2$$

$$= (|z_1| \cdot |z_2|)^2 \text{ from (b)}$$

$$\Rightarrow |z_1 \cdot z_2| = |z_1| \cdot |z_2|$$

$$\text{proof (d) } \left| \frac{z_1}{z_2} \right|^2 = \frac{z_1(\bar{z}_1)}{z_2(\bar{z}_2)} = \frac{|z_1|^2}{|z_2|^2} = \left(\frac{|z_1|}{|z_2|} \right)^2$$

$$\Rightarrow \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}.$$

The others are left for the reader.

1.5. Additive and multiplicative inverses

Let $z = x + iy$ be a complex number, then

i) its additive inverse denoted by $(-z)$ is given by: $-z = -(x + iy) = -x - iy$.

ii) its multiplicative inverse denoted by z^{-1} is given by: $z^{-1} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}.$

Example 6: Find the additive and the multiplicative inverse of $z = 3+4i$.

Solution : $z = 3 + 4i$

i) $-z = -3 - 4i$

ii) $z^{-1} = \frac{1}{3+4i} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3}{25} - \frac{4i}{25}$.

Exercise

1. Verify that

a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$

b) $(2, -3)(-2, 1) = (-1, 8)$

c) $(3, 1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right) = (2, 1)$

d) $(2 + 3i)^2 - (3i - 6) = 1 + 9i$

2. Show that

a) $\operatorname{Re}(iz) = -\operatorname{Im}(z)$

b) $\operatorname{Im}(iz) = \operatorname{Re}(z)$

c) $(z+1)^2 = z^2 + 2z + 1$

3. Do the following operations and simplify your answer.

a) $\frac{1+2i}{3-4i} + \frac{2-i}{5i}$

b) $\frac{5i}{(1-i)(2-i)(3-i)}$

c) $(1-i)^3$

4. Locate the complex numbers $z_1 + z_2$ and $z_1 - z_2$, as vectors where

a) $z_1 = 2i, z_2 = \frac{2}{3} - i$

b) $z_1 = (-\sqrt{3}, 1), z_2 = (\sqrt{3}, 0)$

c) $z_1 = (-3, 1), z_2 = (1, 4)$

d) $z_1 = a + ib, z_2 = a - ib$

5. Sketch the following set of points determined by the condition given below:

a) $|z - 1 + i| = 1$

b) $|z + i| \leq 3$

c) $|z - 4i| \geq 4$

6. Using properties of conjugate and modulus, show that

a) $\overline{\overline{z} + 3i} = z - 3i$

b) $\overline{iz} = -i\overline{z}$

c) $\overline{(2+i)^2} = 3 - 4i$

7. Show that $(-1+i)^7 = 8(-1-i)$.

8. Using mathematical induction, show that (when $n = 2, 3, \dots$)

a) $\overline{z_1 + z_2 + \dots + z_n} = \overline{z_1} + \overline{z_2} + \dots + \overline{z_n}$

b) $\overline{z_1 z_2 \dots z_n} = \overline{z_1} \overline{z_2} \dots \overline{z_n}$

9. Show that the equation $|z - z_0| = r$ which is a circle of radius r centered at z_0 can be

written as $|z|^2 - 2\operatorname{Re}(z\bar{z}_0) + |z_0|^2 = r^2$.

1.6 Argument (Amplitude) of a complex number

Definition 2.15

Argument of a complex number $z = x+iy$ is the angle formed by the complex number $z = x+iy$ with the positive x-axis. The argument of a complex number $z = x+iy$ is denoted by **argz** and is given by $\arg(z) = \tan^{-1}(y/x)$.

The particular argument of z that lies in the range $-\pi < \theta \leq \pi$ is called the principal argument of z and is denoted by **Argz**.

Notes : i) $\operatorname{Arg} z \in (-\pi, \pi]$

ii) If $0 \leq \operatorname{Arg} z \leq \pi$, move counterclockwise direction, if not move the other direction.

Example 2.50: Find the principal argument of the following complex numbers:

a) $z = 1 + i$

b) $z = -2 + 2\sqrt{3}i$

c) $z = -\sqrt{3} - i$

Sol : a) $z = 1 + i$

$$\operatorname{Arg} z = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

b) $z = -2 + 2\sqrt{3}i$

$$\operatorname{Arg} z = \tan^{-1}\left(\frac{2\sqrt{3}}{-2}\right) = \tan^{-1}(-\sqrt{3}) = \frac{2\pi}{3}$$

c) $z = -\sqrt{3} - i$

$$\operatorname{Arg} z = \tan^{-1}\left(\frac{-1}{-\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{-5\pi}{6}$$

1.6.1. Properties of Arguments

$$i) \operatorname{Arg}(z_1 \cdot z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \quad ii) \operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg} z_1 - \operatorname{Arg} z_2$$

Example 4: Find the principal argument of a) $(1+i)(-1-i)$ b) $\left(\frac{-2+2i}{1-i}\right)$

Solution :

$$a) \operatorname{Arg} (1+i)(-1-i) = \operatorname{Arg} (1+i) + \operatorname{Arg} (-1-i) = \frac{\pi}{4} + \left(-\frac{3\pi}{4}\right) = \underline{\underline{-\frac{\pi}{2}}}$$

$$b) \operatorname{Arg} \left(\frac{-2+2i}{1-i}\right) = \operatorname{Arg} (-2+2i) - \operatorname{Arg} (1-i) = \frac{3\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{4\pi}{4} = \underline{\underline{\pi}}$$

1.6. Polar form of a complex number

Definition 2.16

Let r and θ be polar coordinates of the point (x, y) of the complex number $z = x+iy$. Since $x = r \cos \theta$ and $y = r \sin \theta$, then the complex number can be written as : $\underline{\underline{z = r(\cos \theta + i \sin \theta)}}$ which is called polar form, where r is modulus of z and θ is principal argument of z .

Example 2.52: Express the following complex numbers in polar form:

$$a) z = 1+i$$

$$\text{solution : } r = \sqrt{2} \text{ and } \theta = \tan^{-1}(1) = \frac{\pi}{4}. \text{ Thus, } z = \sqrt{2}(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}).$$

$$b) z = 3-3i$$

$$\text{solution : } r = \sqrt{18} \text{ and } \theta = \tan^{-1}(-1) = -\frac{\pi}{4}.$$

$$\text{Thus, } z = \sqrt{18}(\cos -\frac{\pi}{4} + i \sin -\frac{\pi}{4}) = \sqrt{18}(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}).$$

Multiplication and division in polar forms

If $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, then

$$a) z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)) \quad b) \frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

Proof:

$$\begin{aligned}
 a) \quad z_1 \cdot z_2 &= r_1(\cos \theta_1 + i \sin \theta_1) \cdot r_2(\cos \theta_2 + i \sin \theta_2) \\
 &= r_1 \cdot r_2 [\cos \theta_1 (\cos \theta_2 + i \sin \theta_2) + i \sin \theta_1 (\cos \theta_2 + i \sin \theta_2)] \\
 &= r_1 \cdot r_2 [\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2] \\
 &= r_1 \cdot r_2 [\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 + i (\cos \theta_1 \sin \theta_2 + \sin \theta_1 \cos \theta_2)] \\
 &= \underline{\underline{r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]}}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad \frac{z_1}{z_2} &= \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)} = \frac{r_1}{r_2} \left(\frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \right) \\
 &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 + i \sin \theta_1}{\cos \theta_2 + i \sin \theta_2} \right) \cdot \frac{\cos \theta_2 - i \sin \theta_2}{\cos \theta_2 - i \sin \theta_2} \\
 &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 \cos \theta_2 - i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}{\cos^2 \theta_2 + \sin^2 \theta_2} \right) \\
 &= \frac{r_1}{r_2} \left(\frac{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + i (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2)}{1} \right) \\
 &= \underline{\underline{\frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]}}
 \end{aligned}$$

Example 2.53: If $z_1 = 6(\cos \pi/2 + i \sin \pi/2)$ and $z_2 = 2(\cos \pi/3 + i \sin \pi/3)$, then find

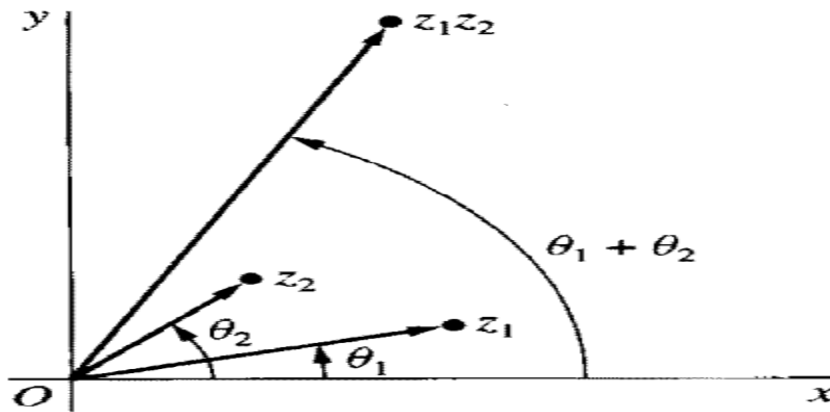
$$a) \quad z_1 \cdot z_2 \qquad b) \quad \frac{z_1}{z_2}$$

Solution :

$$\begin{aligned}
 a) \quad z_1 \cdot z_2 &= 6 \cdot 2 [\cos(\pi/2 + \pi/3) + i \sin(\pi/2 + \pi/3)] \\
 &= \underline{\underline{12 [\cos 5\pi/6 + i \sin 5\pi/6]}} \\
 b) \quad \frac{z_1}{z_2} &= \frac{6}{2} [\cos(\pi/2 - \pi/3) + i \sin(\pi/2 - \pi/3)] \\
 &= \underline{\underline{3 [\cos \pi/6 + i \sin \pi/6]}}
 \end{aligned}$$

1.7.1 Argument of a product

The argument of the product of two complex numbers is the sum of their arguments.



Proof:

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\begin{aligned}
 \text{Now } z_1 \cdot z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2) \\
 &= r_1 \cdot r_2 [\cos\theta_1(\cos\theta_2 + i\sin\theta_2) + i\sin\theta_1(\cos\theta_2 + i\sin\theta_2)] \\
 &= r_1 \cdot r_2 [\cos\theta_1 \cos\theta_2 + i\cos\theta_1 \sin\theta_2 + i\sin\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2] \\
 &= r_1 \cdot r_2 [\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)] \\
 &= \underline{\underline{r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]}} \\
 \Rightarrow \arg(z_1 \cdot z_2) &= \tan^{-1} \left(\frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} \right) \\
 &= \tan^{-1}(\tan(\theta_1 + \theta_2)) \\
 &= \underline{\underline{\theta_1 + \theta_2}}
 \end{aligned}$$

1.7.2. Argument of a quotient

The argument of the quotient of two complex numbers is the difference of their arguments.

Proof:

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1 \cdot z_2^{-1}) = \arg(z_1) + \arg(z_2^{-1}) = \arg(z_1) + -(\arg z_2) = \arg(z_1) - \arg(z_2)$$

Example 2.54: $\arg\left(\frac{-4}{1 + \sqrt{3}i}\right) = \arg(-4) - \arg(1 + \sqrt{3}i) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

De Moivre's Formula

recall the product: $z_1 \cdot z_2 = r_1 \cdot r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$.

Similarly, we get $z_1 \cdot z_2 \cdots z_n = r_1 \cdot r_2 \cdots r_n [\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin(\theta_1 + \theta_2 + \cdots + \theta_n)]$.

Now we can generalize that $z^n = r^n [\cos(\theta + \theta + \cdots + \theta) + i \sin(\theta + \theta + \cdots + \theta)]$
 $= \underline{\underline{r^n [\cos n\theta + i \sin n\theta]}}$ which is called De Moivre's formula.

Example 2.55: Express $(2 + 2i)^{100}$ in polar form.

Solution: Let $z = 2 + 2i$. Then, $r = \sqrt{8}$, $\theta = \pi/4$ and hence

$$\begin{aligned} (2 + 2i)^{100} &= z^{100} \\ &= \sqrt{8}^{100} [\cos 100(\pi/4) + i \sin 100(\pi/4)] \\ &= \underline{\underline{8^{50} [\cos 25\pi + i \sin 25\pi]}}. \end{aligned}$$

Example 2.56: Express $(\sqrt{3} + i)^{60}$ in polar form.

Solution: Let $z = \sqrt{3} + i$. Then, $r = 2$, $\theta = \pi/6$.

$$\begin{aligned} \therefore (\sqrt{3} + i)^{60} &= z^{60} \\ &= 2^{60} [\cos 60(\pi/6) + i \sin 60(\pi/6)] \\ &= \underline{\underline{2^{60} [\cos 10\pi + i \sin 10\pi]}}. \end{aligned}$$

Euler's formula

The complex number $z = r(\cos \theta + i \sin \theta)$ can be written in exponential form as: $z = re^{i\theta}$ which is called Euler's formula.

Note: $z^n = r^n (\cos n\theta + i \sin n\theta) = r^n e^{i(n\theta)}$

Example 2.57: Express the complex number $z = 1 + i$ using Euler's formula.

Solution : $z = 1 + i$

Now $r = \sqrt{2}$ & $\theta = \pi/4 \Rightarrow z = re^{i\theta} = \underline{\underline{\sqrt{2}e^{\pi i/4}}}$

Example 2.58: Express the complex number $z = 1 + \sqrt{3}i$ using Euler's formula.

Solution : $z = 1 + \sqrt{3}i$

Now $r = 2$ & $\theta = \pi/3 \Rightarrow z = re^{i\theta} = \underline{\underline{2e^{\pi i/3}}}$

Example 2.59: Express the complex number $z = (\sqrt{3} + i)^7$ using Euler's formula.

Solution : $z = (\sqrt{3} + i)^7$

Now $r = 2$, $\theta = \pi/6 \Rightarrow (\sqrt{3} + i)^7 = 2^7 e^{\frac{7\pi i}{6}} = 128e^{\frac{7\pi i}{6}}$

1. 8 Extraction of roots

Suppose $z_o = r_o e^{i\theta_o}$ is the n^{th} root of a non-zero complex number $z = re^{i\theta}$, where $n \geq 2$.

Then, $z_o^n = z$, which implies that $r_o^n e^{in\theta_o} = re^{i\theta}$

$$\Rightarrow r_o^n = r \text{ \& } n\theta_o = \theta + 2k\pi, \quad k = 0, 1, 2, \dots, (n-1).$$

$$\Rightarrow r_o = (r)^{1/n} \text{ \& } \theta_o = \frac{\theta}{n} + \frac{2k\pi}{n}$$

$\therefore z_o = \underline{\underline{(r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})})}}$ which is the n^{th} root of z , where $n = 2, 3, \dots$ and $k = 0, 1, 2, \dots, (n-1)$

or we can denote it by C_k as : $C_k = \underline{\underline{(r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})})}}$, $k = 0, 1, 2, \dots, (n-1)$

Example 2.61: Find the cube roots of the complex number $z = 8i$.

Solution : We have $z = 8i$. Here $r = 8$, $\theta = \pi/2$, $n = 3$, $k = 0, 1, 2$.

Hence, $C_k = (r)^{1/n} (e^{i(\frac{\theta}{n} + \frac{2k\pi}{n})})$

$$\Rightarrow C_k = (8)^{1/3} (e^{i(\frac{\pi/2}{3} + \frac{2k\pi}{3})})$$

$$\Rightarrow C_k = 2 (e^{i(\pi/6 + \frac{2k\pi}{3})})$$

$$i) \text{ If } k=0, C_0 = 2(e^{i(\pi/6)}) = (\cos \pi/6 + i \sin \pi/6) = 2(\frac{\sqrt{3}}{2} + \frac{i}{2}) = \underline{\underline{\sqrt{3} + i}}$$

$$ii) \text{ If } k=1, C_1 = 2(e^{i(\pi/6 + 2\pi/3)}) = 2(\cos 5\pi/6 + i \sin 5\pi/6) = 2(\frac{-\sqrt{3}}{2} + \frac{i}{2}) = \underline{\underline{-\sqrt{3} + i}}$$

$$iii) \text{ If } k=2, C_2 = 2(e^{i(\pi/6 + 4\pi/3)}) = 2(\cos 3\pi/2 + i \sin 3\pi/2) = 2(0 - i) = \underline{\underline{-2i}}$$

$\therefore$ The cube roots of $8i$ are $C_0 = \sqrt{3} + i$, $C_1 = -\sqrt{3} + i$ & $C_2 = -2i$.

Exercise 2.4

1. Find the argument of the following complex numbers:

$$a) z = \frac{3i}{-1-i}$$

$$b) z = (\sqrt{3} - i)^6$$

2. Show that a) $|e^{i\theta}| = 1$

$$b) \overline{e^{i\theta}} = e^{-i\theta}$$

3. Using mathematical induction, show that $e^{i\theta_1} \cdot e^{i\theta_2} \cdot \dots \cdot e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \dots + \theta_n)}$, $n = 2, 3, \dots$

4. Show that a) $\cos 3\theta = \cos^3 \theta - 3\cos \theta \sin^2 \theta$ b) $\sin 3\theta = 3\cos^2 \theta \sin \theta - \sin^3 \theta$

5. Show that $1 + z + z^2 + \dots + z^n = \frac{1 - z^{n+1}}{1 - z}$, for $z \neq 1$.

6. Find the square roots of $z = 9i$

7. Find the cube roots of $z = -8i$

8. Solve the following equations:

$$a) z^{3/2} = 8i \quad b) z^2 + 4i = 0 \quad c) z^2 - 4i = 0$$